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Marcinkiewicz multiplier theorem and the Sunouchi operator for Ciesielski–Fourier series

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Abstract

Some classical results due to Marcinkiewicz, Littlewood and Paley are proved for the Ciesielski–Fourier series. The Marcinkiewicz multiplier theorem is obtained for L_p spaces and extended to Hardy spaces. The boundedness of the Sunouchi operator on L_p and Hardy spaces is also investigated.

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1. Introduction

For trigonometric and Walsh–Fourier series the partial sum operators are bounded on L_p $(1 < p < \infty)$ spaces. A vector-valued version of this theorem is due to Marcinkiewicz and Zygmund for trigonometric Fourier series (see e.g. [40, II. p. 225]), to Sunouchi [33] for Walsh–Fourier series and to Young [39] for Vilenkin–Fourier series. By the Littlewood–Paley theory the L_p norm $(1 < p < \infty)$ of the square function of f is equivalent to the L_p norm of f (for the Walsh system see e.g. [21], for the trigonometric series, see [40, II. p. 224] or [11]).

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Marcinkiewicz (see e.g. [40, II. p. 232]) gave a sufficient condition for a multiplier operator of the trigonometric Fourier series to be bounded on L_p $(1<p<\infty)$ spaces. The same theorem is proved by Young [39] for Vilenkin–Fourier series. Hörmander [16] generalized the Marcinkiewicz condition and theorem. Under some Hörmander-type conditions the boundedness of the multiplier operator was proved also on the Hardy spaces H_p (for trigonometric Fourier series see [1,8,20], for Walsh- and Vilenkin–Fourier series see [7,17–19]).

In this paper we extend these results to Ciesielski–Fourier series, which are generalizations of the Walsh–Fourier series. The Ciesielski systems can be obtained from the spline systems of order (m, k) in the same way as the Walsh system arises from the Haar system (see [4–6]). The Marcinkiewicz multiplier theorem is extended in another way to Hardy spaces, which is even new for the Walsh system. A sufficient condition is given for the multiplier operator to be bounded from the H_p Hardy space to L_p, where $p_0 < p \leqslant 1$ and p_0 is depending on the multiplier and on m and k. It is also proved that the Littlewood–Paley-type square function is bounded from H_p to L_p $(p_0 < p \leqslant 1)$.

For Walsh–Fourier series Sunouchi [32,33] introduced an operator and verified that it is bounded on L_p $(1 < p < \infty)$ spaces. This operator was used to prove some strong summability results of Fourier series. The analogous statement fails to hold for $p = 1$ (see [34]). The corresponding theorem for trigonometric Fourier series can be found in [40, II. p. 224]. Many authors have investigated the Sunouchi operator U (e.g. [10,14,15,24,25,27–29]) for Walsh-, Walsh–Kaczmarz and Vilenkin systems. Simon [24] verified that U is bounded from H_p to L_p for $p = 1$. This result was extended recently to all $0 < p \leqslant 1$ by Weisz [37] and Simon [25]. By using our multiplier theorems mentioned above, in the last section these results will be generalized for Ciesielski–Fourier series.

2. Ciesielski systems

We consider the unit interval $[0, 1)$ and the Lebesgue measure λ on it. We also use the notation $|I|$ for the Lebesgue measure of the set I. For brevity we write L_p instead of the real $L_p([0, 1), \lambda)$ space while the norm (or quasi-norm) of this space is defined by $\|f\|_p := (\int_{[0,1)} |f|^p \, d\lambda)^{1/p}$ $(0 < p \leqslant \infty)$. The space l_p consists of those sequences $b = (b_n, n \in \mathbb{N})$ of real numbers for which

$$\|b\|_{l_p} := \left(\sum_{n\in\mathbb{N}} |b_n|^p \right)^{1/p} < \infty$$

while $L_p(l_r)$ $(1 \leqslant p, r < \infty)$ consists of all sequences $f := (f_n, n \in \mathbb{N})$ of functions for which

$$\|f\|_{L_p(l_r)} := \left\| \left(\sum_{n\in\mathbb{N}} |f_n|^r \right)^{1/r} \right\|_p < \infty.$$

First we define the Walsh system. Let

$$r(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}), \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1. The *Rademacher system* $(r_n, n \in \mathbb{N})$ is defined by

$$r_n(x) := r(2^n x) \quad (x \in [0, 1), n \in \mathbb{N}).$$

The *Walsh functions* are given by

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \quad (x \in [0, 1), n \in \mathbb{N})$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$, $(n_k = 0$ or $n_k = 1)$. It is known that $w_n(t)w_n(x) = w_n(x \dot{+} t)$ $(n \in \mathbb{N}, t, x \in [0, 1))$, where the dyadic addition $\dot{+}$ is defined e.g. in [23].

Next we introduce the spline systems as in Ciesielski [5]. Let us denote by D the differentiation operator and define the integration operators

$$Gf(t) := \int_0^t f \, d\lambda, \quad Hf(t) := \int_t^1 f \, d\lambda.$$

Define the χ_n, $n = 1, 2, \ldots$, *Haar system* by $\chi_1 := 1$ and

$$\chi_{2^n+k}(x) := \begin{cases} 2^{n/2} & \text{if } x \in ((2k-2)2^{-n-1}, (2k-1)2^{-n-1}), \\ -2^{n/2} & \text{if } x \in ((2k-1)2^{-n-1}, (2k)2^{-n-1}), \\ 0 & \text{otherwise} \end{cases}$$

for $n, k \in \mathbb{N}$, $0 < k \leqslant 2^n$, $x \in [0, 1)$.

Let $m \geqslant -1$ be a fixed integer. Applying the Schmidt orthonormalization to the linearly independent functions

$$1, t, \ldots, t^{m+1}, G^{m+1}\chi_n(t), \quad n \geqslant 2,$$

we get the *spline system* $(f_n^{(m)}, n \geqslant -m)$ of order m. For $0 \leqslant k \leqslant m+1$ and $n \geqslant k - m$ define the splines

$$f_n^{(m,k)} := D^k f_n^{(m)}, \quad g_n^{(m,k)} := H^k f_n^{(m)}$$

of order (m, k). Let us normalize these functions and introduce a more unified notation,

$$h_n^{(m,k)} := \begin{cases} f_n^{(m,k)} \| f_n^{(m,k)} \|_2^{-1} & \text{for } 0 \leqslant k \leqslant m+1, \\ g_n^{(m,-k)} \| f_n^{(m,-k)} \|_2 & \text{for } 0 \leqslant -k \leqslant m+1. \end{cases}$$

We get the Haar system if $m = -1$, $k = 0$ and the *Franklin system* if $m = 0$, $k = 0$. The systems $(h_i^{(m,k)}, i \geqslant |k| - m)$ and $(h_j^{(m,-k)}, j \geqslant |k| - m)$ are biorthogonal, i.e.

$$(h_i^{(m,k)}, h_j^{(m,-k)}) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where (f, g) denotes the usual scalar product $\int_{[0,1)} fg \, d\lambda$.

It is proved in Ciesielski [4,5] that

$$|D^N h_{2^\mu+\nu}^{(m,k)}(t)| \leqslant C 2^{(N+1/2)\mu} q^{2^\mu|t-\nu/2^\mu|}, \qquad (1)$$

where $m \geqslant -1$, $|k| \leqslant m+1$, $k+N \leqslant m+1$, $\mu \in \mathbb{N}$ and $\nu = 1, \ldots, 2^\mu$.

In this paper, the constants C and q are depending only on m and the constants C_p are depending only on p and m. The constants C, q and C_p may denote different constants in different contexts, however, q denote constants for which $0 < q < 1$.

Starting with the spline system $(h_n^{(m,k)}, n \geqslant |k| - m)$ we define the *Ciesielski system* $(c_n^{(m,k)}, n \geqslant |k| - m)$ in the same way as the Walsh system arises from the Haar system, namely,

$$c_n^{(m,k)} := h_n^{(m,k)} \quad (n = |k| - m, \ldots, 1)$$

and

$$c_{2^\nu+i}^{(m,k)} := \sum_{j=1}^{2^\nu} A_{i,j}^{(\nu)} h_{2^\nu+j}^{(m,k)} \quad (1 \leqslant i \leqslant 2^\nu).$$

We get immediately that

$$h_{2^\nu+j}^{(m,k)} := \sum_{i=1}^{2^\nu} A_{i,j}^{(\nu)} c_{2^\nu+i}^{(m,k)} \quad (1 \leqslant j \leqslant 2^\nu).$$

As mentioned before,

$$c_n^{(-1,0)} = w_{n-1} \quad (n \geqslant 1)$$

is the usual Walsh system. One can show (see [23] or [6]) that

$$A_{i,j}^{(\nu)} = A_{j,i}^{(\nu)} = 2^{-\nu/2} w_{i-1}\left(\frac{2j-1}{2^{\nu+1}}\right). \qquad (2)$$

The system $(c_n^{(m,k)})$ is uniformly bounded and it is biorthogonal to $(c_n^{(m,-k)})$ whenever $|k| \leqslant m+1$.

3. Littlewood–Paley-type inequality

The *partial sums* and the *Fejér means* of the Ciesielski–Fourier series are defined by

$$s_n^{(m,k)} f(x) := \sum_{j=|k|-m}^{n} (f, c_j^{(m,k)}) c_j^{(m,-k)}(x) = \int_0^1 D_n^{(m,k)}(t,x) f(t)\, dt,$$

$$\sigma_n^{(m,k)} f(x) := \frac{1}{n} \sum_{j=1}^{n} s_j^{(m,k)}(x) = \int_0^1 K_n^{(m,k)}(t,x) f(t)\, dt,$$

respectively, where $m \geqslant -1$ and $|k| \leqslant m+1$. Here

$$D_n^{(m,k)}(t,x) := \sum_{j=|k|-m}^{n} c_j^{(m,k)}(t) c_j^{(m,-k)}(x),$$

$$K_n^{(m,k)}(t,x) := \frac{1}{n} \sum_{j=1}^{n} D_j^{(m,k)}(t,x)$$

are the *Dirichlet* and *Fejér kernels*.

The Walsh–Dirichlet and Walsh–Fejér kernels $D_n^{(-1,0)}$ and $K_n^{(-1,0)}$ are denoted by D_n and K_n, respectively. It is known [23] that $D_n(t,x) = D_n(t \dot{+} x)$, $K_n(t,x) = K_n(t \dot{+} x)$ and

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if } x \in [0, 2^{-n}), \\ 0 & \text{if } x \in [2^{-n}, 1), \end{cases} \tag{3}$$

$$|K_n(x)| \leqslant 2 \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} D_{2^i}(x \dot{+} 2^{-j-1}), \tag{4}$$

where $x \in [0,1)$, $2^{N-1} \leqslant n < 2^N$ and

$$K_{2^n}(x) = C \sum_{j=0}^{n} 2^{j-n} D_{2^n}(x \dot{+} 2^{-j-1}). \tag{5}$$

Ciesielski [5] proved that

$$\| \sup_{n \in \mathbb{N}} |s_n^{(m,k)} f| \|_p \leqslant C_p \|f\|_p \quad (1 < p < \infty), \tag{6}$$

where $|k| \leqslant m+1$. In this section we will show a vector-valued version of this inequality.

Let us first introduce the *Hardy–Littlewood maximal function*. For $f \in L_1$ let

$$Mf(x) := \sup_{x \in I} \frac{1}{|I|} \int_I |f| \, d\lambda \quad (x \in [0,1)),$$

where the supremum is taken over all intervals containing x. It is known that (see [30, p. 51])

$$\int_0^1 \left(\sum_{i=0}^{\infty} |Mf_i|^r \right)^{p/r} d\lambda \leqslant C_{p,r} \int_0^1 \left(\sum_{i=0}^{\infty} |f_i|^r \right)^{p/r} d\lambda \tag{7}$$

for $f = (f_i, i \in \mathbb{N}) \in L_p(l_r)$ $(1 < p, r < \infty)$.

The vector-valued Calderon–Zygmund decomposition lemma (see e.g. [33]) can be used to prove the next weak type inequality (cf. [38, p. 44]). If I is an interval then let rI be the interval having the same center as I and length $|rI| = r|I|$ $(r \in \mathbb{N})$.

Theorem 1. *Suppose that the sublinear operator V is bounded from $L_{p_1}(l_r)$ to $L_{p_1}(l_r)$ for some $1 < p_1, r \leqslant \infty$ and*

$$\int_{[0,1) \setminus 2I} \|Vf\|_{l_r} \, d\lambda \leqslant C \|f\|_{L_1(l_r)}$$

for all $f \in L_1(l_r)$ *and intervals I which satisfy*

$$\operatorname{supp} f \subset I \quad \textit{and} \quad \int_0^1 f\, d\lambda = 0. \tag{8}$$

Then the operator V is of weak type $(L_1(l_r), L_1(l_r))$, *i.e.*

$$\sup_{\rho>0} \rho\lambda(\|Vf\|_{l_r} > \rho) \leqslant C\|f\|_{L_1(l_r)} \quad (f \in L_1(l_r)).$$

Let us introduce the following operator:

$$P_n^{(m,k,m',k')} f := \sum_{j=(|k|-m)\vee(|k'|-m')}^{n} (f, h_j^{(m,k)}) h_j^{(m',-k')},$$

where $m \geqslant -1$, $m' \geqslant -1$, $|k| \leqslant m+1$, $|k'| \leqslant m'+1$. If $m = m'$ and $k = k'$ then we write $P_n^{(m,k,m',k')} = P_n^{(m,k)}$. If $1 < p < \infty$ then

$$\|P_n^{(m,k,m',k')} f\|_p \leqslant C_p \|f\|_p \quad (f \in L_p) \tag{9}$$

uniformly in $n \in \mathbb{N}$ (see [5]).

The following lemma can be found in Weisz [38].

Lemma 1. *Suppose that* $m \geqslant -1$, $m' \geqslant -1$, $|k| \leqslant m+1$, $|k'| \leqslant m'+1$ *and* $k + N \leqslant m+1$. *Then*

$$\sum_{j=0}^{\infty} \sum_{2^j < i \leqslant 2^{j+1}} |(D^N h_i^{(m,k)}(t)) h_i^{(m',-k')}(x)| \leqslant C|x-t|^{-(N+1)}$$

and for all $K \in \mathbb{N}$,

$$\sum_{j=K}^{\infty} \sum_{2^j < i \leqslant 2^{j+1}} |h_i^{(m,k)}(t) h_i^{(m',-k')}(x)| \leqslant C 2^{-K} |x-t|^{-2}.$$

The corresponding result to (7) for the operators $P_n^{(m,k,m',k')}$ reads as follows.

Theorem 2. *Assume that* $m \geqslant -1$, $m' \geqslant -1$, $|k| \leqslant m+1$, $|k'| \leqslant m'+1$ *and* $f = (f_i, i \in \mathbb{N}) \in L_p(l_r)$ $(1 < p, r < \infty)$. *If* $n(i)$ *is an arbitrary natural number for each* $i \in \mathbb{N}$ *then*

$$\int_0^1 \left(\sum_{i=0}^{\infty} |P_{n(i)}^{(m,k,m',k')} f_i|^r \right)^{p/r} d\lambda \leqslant C_{p,r} \int_0^1 \left(\sum_{i=0}^{\infty} |f_i|^r \right)^{p/r} d\lambda. \tag{10}$$

Proof. Observe that (10) for $p = r$ follows from (9). Let $g \in L_1$ with support I satisfying $\int_0^1 g\, d\lambda = 0$ (see (8)). Then

$$P_n^{(m,k,m',k')} g(x) = \int_I g(t) \sum_{j=(|k|-m)\vee(|k'|-m')}^{n} h_j^{(m,k)}(t) h_j^{(m',-k')}(x)\, dt$$

$$
\begin{aligned}
&= \int_I g(t) \sum_{j=(|k|-m)\vee(|k'|-m')}^{1} h_j^{(m,k)}(t) h_j^{(m',-k')}(x)\, dt \\
&\quad + \int_I g(t) \sum_{j=2}^{n} h_j^{(m,k)}(t) h_j^{(m',-k')}(x)\, dt \\
&=: A_1(x) + A_2(x).
\end{aligned}
$$

Since $h_j^{(m,k)} \in L_\infty$ $(j \leqslant 1)$,

$$
|A_1(x)| \leqslant \int_I |g(t)|\, dt.
$$

If $k \leqslant m$ then

$$
A_2(x) = \int_I g(t) \sum_{j=2}^{n} (h_j^{(m,k)}(t) - h_j^{(m,k)}(t_0)) h_j^{(m',-k')}(x)\, dt
$$

where t_0 denotes the center of I. By Lagrange's theorem and Lemma 1,

$$
\begin{aligned}
|A_2(x)| &\leqslant |I| \int_I |g(t)| \sum_{j=2}^{n} |Dh_j^{(m,k)}(\xi)||h_j^{(m',-k')}(x)|\, dt \\
&\leqslant |I| \int_I |g(t)| \sum_{j=0}^{\infty} \sum_{2^j < i \leqslant 2^{j+1}} |Dh_j^{(m,k)}(\xi)||h_j^{(m',-k')}(x)|\, dt \\
&\leqslant C|I| \int_I |g(t)||x - t_0|^{-2}\, dt
\end{aligned}
$$

if $\xi \in I$ and $x \notin 2I$.

If $k = m + 1$ and $j \leqslant 2^K$ then $h_j^{(m,k)}$ is constant on I, where we may suppose that I is dyadic and $|I| = 2^{-K}$. Thus $P_n^{(m,k,m',k')} g = 0$ for $n \leqslant 2^K$. If $n > 2^K$, then

$$
\begin{aligned}
|A_2(x)| &\leqslant \int_I |g(t)| \sum_{j=2^K+1}^{n} |h_j^{(m,k)}(t)||h_j^{(m',-k')}(x)|\, dt \\
&\leqslant \int_I |g(t)| \sum_{j=K}^{\infty} \sum_{2^j < i \leqslant 2^{j+1}} |h_j^{(m,k)}(t)||h_j^{(m',-k')}(x)|\, dt \\
&\leqslant C|I| \int_I |g(t)||x - t_0|^{-2}\, dt.
\end{aligned}
$$

Assume that $f \in L_1(l_r)$ has support I and satisfies (8). From the above inequalities it follows that

$$
\begin{aligned}
\left(\sum_{i=0}^{\infty} |P_{n(i)}^{(m,k,m',k')} f_i(x)|^r \right)^{1/r} &\leqslant C|I||x - t_0|^{-2} \left(\sum_{i=0}^{\infty} \left(\int_0^1 |f_i|\, d\lambda \right)^r \right)^{1/r} \\
&\leqslant C|I||x - t_0|^{-2} \int_0^1 \left(\sum_{i=0}^{\infty} |f_i|^r \right)^{1/r} d\lambda
\end{aligned}
$$

and

$$\int_{(2I)^c} \left(\sum_{i=0}^{\infty} |P_{n(i)}^{(m,k,m',k')} f_i(x)|^r \right)^{1/r} dx \leqslant C \int_0^1 \left(\sum_{i=0}^{\infty} |f_i|^r \right)^{1/r} d\lambda.$$

Now Theorem 1 implies

$$\sup_{\rho>0} \rho\lambda \left(\left(\sum_{i=0}^{\infty} |P_{n(i)}^{(m,k,m',k')} f_i(x)|^r \right)^{1/r} > \rho \right) \leqslant C \|f\|_{L_1(l_r)} \quad (f \in L_1(l_r)).$$

Inequality (10) for $1 < p < r$ follows easily by interpolation (see e.g. [3] or [2]). For $p > r$ it can be obtained by the usual duality argument. □

Note that Theorem 2 could also be proved by using the corresponding result for the Haar system and by the equivalence of the spline system and Haar system in $L_p(l_r)$. This equivalence can be found in [12,13]. Actually, they proved the equivalence in more general *UMD* spaces. This is a general and complicated result, so for the sake of completeness, we presented a simpler proof of Theorem 2.

The following result was proved by Marcinkievicz and Zygmund for trigonometric Fourier series (see e.g. [40, II. p. 225]) and by Sunouchi [33] for Walsh–Fourier series.

Theorem 3. *Assume that* $m \geqslant -1$, $|k| \leqslant m+1$ *and* $f = (f_i, i \in \mathbb{N}) \in L_p(l_r)$ $(1 < p, r < \infty)$. *If* $n(i)$ *is an arbitrary natural number for each* $i \in \mathbb{N}$ *then*

$$\int_0^1 \left(\sum_{i=0}^{\infty} |s_{n(i)}^{(m,k)} f_i|^r \right)^{p/r} d\lambda \leqslant C_{p,r} \int_0^1 \left(\sum_{i=0}^{\infty} |f_i|^r \right)^{p/r} d\lambda. \tag{11}$$

Proof. If every $n(i)$ is a 2-power, i.e. $n(i) = 2^{n_1(i)}$ then (11) follows from Theorem 2, because it is easy to see that $s_{n(i)}^{(m,k)} = P_{n(i)}^{(m,k)}$.

Set

$$G_\mu^{(m,k)}(t,s) := 2^{\mu/2} r_\mu(s) h_{2^\mu+v}^{(m,k)}(t) \quad \text{if } \frac{v-1}{2^\mu} \leqslant s < \frac{v}{2^\mu} \tag{12}$$

$(1 \leqslant v \leqslant 2^\mu)$. Then, by (2), it is easy to see that

$$c_{2^\mu+v}^{(m,k)}(t) = \int_0^1 c_{2^\mu+v}^{(-1,0)}(s) G_\mu^{(m,k)}(t,s)\, ds \tag{13}$$

where $\mu \in \mathbb{N}$ and $1 \leqslant v \leqslant 2^\mu$ (see also [22] and [6]). Let us write $n \in \mathbb{N}$ in the form $n = 2^i + j$ with $1 \leqslant j \leqslant 2^i$. For $g \in L_1$,

$$s_n^{(m,k)} g = s_{2^i}^{(m,k)} g + \left(s_{2^i+j}^{(m,k)} g - s_{2^i}^{(m,k)} g \right).$$

Therefore

$$
\begin{aligned}
s_{2^i+j}^{(m,k)} g(t) - s_{2^i}^{(m,k)} g(t) &= \sum_{v=1}^{j} (g, c_{2^i+v}^{(m,k)}) c_{2^i+v}^{(m,-k)}(t) \\
&= \int_0^1 G_i^{(m,-k)}(t,s) \sum_{v=1}^{j} (g, c_{2^i+v}^{(m,k)}) c_{2^i+v}^{(-1,0)}(s)\, ds \\
&= \int_0^1 G_i^{(m,-k)}(t,s) \sum_{v=1}^{j} \sum_{\mu=1}^{2^i} A_{v,\mu}^{(i)} (g, h_{2^i+\mu}^{(m,k)}) c_{2^i+v}^{(-1,0)}(s)\, ds.
\end{aligned}
$$

Since $A_{v,\mu}^{(i)} = (h_{2^i+\mu}^{(-1,0)}, c_{2^i+v}^{(-1,0)})$, we have

$$
\begin{aligned}
& s_{2^i+j}^{(m,k)} g(t) - s_{2^i}^{(m,k)} g(t) \\
&\quad = \int_0^1 G_i^{(m,-k)}(t,s) \sum_{v=1}^{j} \left(\sum_{\mu=1}^{2^i} (g, h_{2^i+\mu}^{(m,k)}) h_{2^i+\mu}^{(-1,0)}, c_{2^i+v}^{(-1,0)} \right) c_{2^i+v}^{(-1,0)}(s)\, ds \\
&\quad = \int_0^1 G_i^{(m,-k)}(t,s) \left(s_{2^i+j}^{(-1,0)}(Pg)(s) - s_{2^i}^{(-1,0)}(Pg)(s) \right) ds,
\end{aligned}
$$

where

$$
Pg := P^{(m,k,-1,0)} g := \sum_{n=1}^{\infty} (g, h_n^{(m,k)}) h_n^{(-1,0)}.
$$

Of course, we may suppose that the sum is finite. Ciesielski et al. [5] proved that

$$
\left| \int_0^1 G_i^{(m,-k)}(t,s) h(s)\, ds \right| \leqslant CMh(t) \quad (t \in [0,1), h \in L_1),
$$

which implies

$$
|s_{2^i+j}^{(m,k)} g - s_{2^i}^{(m,k)} g| \leqslant M \left(s_{2^i+j}^{(-1,0)}(Pg) - s_{2^i}^{(-1,0)}(Pg) \right). \tag{14}
$$

Suppose that $n(i) = 2^{n_1(i)} + n(i)^{(1)}$ with $0 \leqslant n(i)^{(1)} < 2^{n_1(i)}$. Taking into account (11) for the Walsh system, Theorem 2 and (7) we obtain

$$
\begin{aligned}
\int_0^1 \left(\sum_{i=0}^{\infty} |s_{n(i)}^{(m,k)} f_i|^r \right)^{p/r} d\lambda &\leqslant \int_0^1 \left(\sum_{i=0}^{\infty} |s_{2^{n_1(i)}}^{(m,k)} f_i|^r \right)^{p/r} d\lambda \\
&\quad + \int_0^1 \left(\sum_{i=0}^{\infty} |M s_{n(i)}^{(-1,0)}(Pf_i)|^r \right)^{p/r} d\lambda \\
&\quad + \int_0^1 \left(\sum_{i=0}^{\infty} |M s_{2^{n_1(i)}}^{(-1,0)}(Pf_i)|^r \right)^{p/r} d\lambda \\
&\leqslant C_p \int_0^1 \left(\sum_{i=0}^{\infty} |f_i|^r \right)^{p/r} d\lambda.
\end{aligned}
$$

This completes the proof of the theorem. □

Now we are going to prove the Littlewood–Paley inequality. Let

$$Q^{(m,k)}f := \left(\sum_{j=|k|-m}^{1} |(f, c_j^{(m,k)})c_j^{(m,-k)}|^2 + \sum_{i=0}^{\infty} |s_{2^{i+1}}^{(m,k)} f - s_{2^i}^{(m,k)} f|^2 \right)^{1/2}$$

be the *square function*. For simplicity, from this time on we suppose that

$$(f, c_j^{(m,k)}) = 0 \quad \text{for } j = |k| - m, \ldots, 1.$$

Of course, all theorems of this paper can similarly be proved without this condition. The following theorem is well known for the Walsh system (see e.g. [21] or in a more general form [35]). For the trigonometric series it can be found in Zygmund [40, II. p. 224] or [11].

Theorem 4. *If* $m \geqslant -1$, $|k| \leqslant m+1$ *and* $f \in L_p$ $(1 < p < \infty)$ *then*

$$C_p \|f\|_p \leqslant \|Q^{(m,k)} f\|_p \leqslant C_p \|f\|_p. \tag{15}$$

This theorem can be proved by applying the unconditionality of $(h_i^{(m,k)}, i \geqslant |k| - m)$ and Khinchine's inequality to $s_{2^{i+1}}^{(m,k)} f - s_{2^i}^{(m,k)} f = P_{2^{i+1}}^{(m,k)} f - P_{2^i}^{(m,k)} f$.

4. Marcinkiewicz multiplier theorem

For a given *multiplier* $\lambda = (\lambda_j, j = 2, \ldots)$ where the λ_j's are real numbers, the *multiplier operators* are defined by

$$T_\lambda^{(m,k)} f := \sum_{j=2}^{\infty} \lambda_j (f, c_j^{(m,k)}) c_j^{(m,-k)}$$

if the sum does exist and by

$$T_{\lambda,n}^{(m,k)} f := \sum_{j=2}^{n} \lambda_j (f, c_j^{(m,k)}) c_j^{(m,-k)} \quad (n \in \mathbb{N}),$$

where $f \in L_1$.

The Marcinkiewicz multiplier theorem is generalized for Ciesielski systems in the next theorem.

Theorem 5. *Assume that* $m \geqslant -1$, $|k| \leqslant m+1$ *and* $f \in L_p$ $(1 < p < \infty)$. *If*

$$|\lambda_i| \leqslant C, \qquad \sum_{j=2^i+1}^{2^{i+1}-1} |\lambda_j - \lambda_{j+1}| \leqslant C \quad (i \in \mathbb{N}) \tag{16}$$

then $T_\lambda^{(m,k)} f \in L_p$ *and*

$$\|T_\lambda^{(m,k)} f\|_p \leqslant C_p \|f\|_p. \tag{17}$$

Proof. Using Theorems 3 and 4 the theorem can be proved in the same way as for the trigonometric system (see [40, II. p. 232]). □

This theorem for Vilenkin–Fourier series is due to Young [39].

Note that with the same conditions $T_\lambda^{(m,k)}$ is not bounded from H_1 to L_1 in general (see [7,8]). Under slightly stronger conditions the Marcinkiewicz multiplier theorem will be extended to Hardy spaces in the next section.

5. Multiplier theorems for Hardy spaces

In order to have a common notation for the dyadic and classical Hardy spaces we define the Poisson kernels $P_t^{(m,k)}$. If $k \leqslant m$ then we introduce $P_t^{(m,k)}$ by

$$P_t^{(m,k)}(x) := \frac{ct}{(t^2 + |x|^2)} \quad (x \in \mathbb{R}, t > 0).$$

If $k = m + 1$ then we define $P_t^{(m,k)}$ as follows. For a fixed $t > 0$ if $n \leqslant t < n + 1$ for some $n \in \mathbb{N}$ then let

$$P_t^{(m,k)}(x) := 1_{[0,2^{-n})}(x) \quad (x \in \mathbb{R}).$$

For a tempered distribution f the *non-tangential maximal function* is defined by

$$f_*^{(m,k)}(x) := \sup_{t>0} |(f * P_t^{(m,k)})(x)| \quad (x \in \mathbb{R})$$

where $*$ denotes the convolution.

For $0 < p < \infty$ the *Hardy space* $H_p^{(m,k)}(\mathbb{R})$ consists of all tempered distributions f for which

$$\|f\|_{H_p^{(m,k)}(\mathbb{R})} := \|f_*^{(m,k)}\|_p < \infty.$$

Now let

$$H_p := H_p^{(m,k)}([0,1)) := \{f \in H_p^{(m,k)}(\mathbb{R}) : \operatorname{supp} f \subset [0,1)\}.$$

Obviously, H_p is the dyadic Hardy space if $k = m + 1$. It is known (see [30]) that the space H_p is equivalent to L_p if $1 < p < \infty$.

A function $a \in L_\infty$ is called a *p-atom* if there exists an interval $I \subset [0, 1)$ such that

(i) $\operatorname{supp} a \subset I$,
(ii) $\|a\|_\infty \leqslant |I|^{-1/p}$,
(iii) $\int_I a(x)x^j\, dx = 0$ where $j \in \mathbb{N}$ and $j \leqslant [1/p - 1]$.

Note that $[x]$ denotes the integer part of $x \in \mathbb{R}$.

In the dyadic case, i.e. if $k = m+1$, we consider only dyadic intervals I and instead of (iii) we assume

(iii′) $\int_I a(x)\,dx = 0$.

Theorem 6 (*Weisz [38]*). *Suppose that the operator V is sublinear and*

$$\int_{[0,1)\setminus 16I} |Va|^p \, d\lambda \leqslant C_p$$

for every p-atom a with support I, where $0 < p \leqslant 1$. If V is bounded from L_{p_1} to L_{p_1} for some $1 < p_1 \leqslant \infty$ then

$$\|Vf\|_p \leqslant C_p \|f\|_{H_p} \quad (f \in H_p).$$

Now we are ready to prove the main theorem of this section. Note that (16) follows from (18).

Theorem 7. *Assume that $m \geqslant -1$, $|k| \leqslant m+1$ and $f \in H_p$ with $1/2 < p < \infty$. If*

$$|\lambda_n| \leqslant C, \qquad \sup_{2^n+1 \leqslant j \leqslant 2^{n+1}-1} j|\lambda_j - \lambda_{j+1}| \leqslant C \quad (n \in \mathbb{N}) \tag{18}$$

and

$$\sum_{j=2^n+1}^{2^{n+1}-2} j|\lambda_j - 2\lambda_{j+1} + \lambda_{j+2}| \leqslant C \quad (n \in \mathbb{N}) \tag{19}$$

then

$$\| \sup_{N \in \mathbb{N}} |T_{\lambda,2^N}^{(m,k)} f| \|_p \leqslant C_p \|f\|_{H_p}. \tag{20}$$

Proof. Since (16) follows from (18), the theorem for $1 < p < \infty$ is a consequence of Theorem 5 and (6).

Suppose that $\frac{1}{2} < p < 1$. Choose a p-atom a with support I and assume that $2^{-K-1} < |I| \leqslant 2^{-K}$ $(K \in \mathbb{N})$ and $x \notin 16I$. Then

$$\begin{aligned} T_{\lambda,2^N}^{(m,k)} a(x) &= \sum_{j=2}^{2^N} \int_I \lambda_j a(t) c_j^{(m,k)}(t)\, dt\, c_j^{(m,-k)}(x) \\ &= \sum_{n=0}^{N-1} \int_I a(t) \sum_{j=2^n+1}^{2^{n+1}} \lambda_j c_j^{(m,k)}(t) c_j^{(m,-k)}(x)\, dt. \end{aligned}$$

By (13),

$$T_{\lambda,2^N}^{(m,k)} a(x) = \sum_{n=0}^{N-1} \int_I a(t) \sum_{j=2^n+1}^{2^{n+1}} \lambda_j$$

$$
\begin{aligned}
&\times \int_0^1 \int_0^1 c_j^{(-1,0)}(s) G_n^{(m,k)}(t,s) c_j^{(-1,0)}(u) G_n^{(m,-k)}(x,u)\, ds\, du\, dt \\
&= \sum_{n=0}^{N-1} \int_I a(t) \int_0^1 \int_0^1 \sum_{j=2^n+1}^{2^{n+1}} \lambda_j c_j^{(-1,0)}(s \dot{+} u) \\
&\times G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)\, ds\, du\, dt.
\end{aligned}
$$

By Abel rearrangement we get that

$$
\begin{aligned}
\sum_{j=2^n+1}^{2^{n+1}} \lambda_j c_j^{(-1,0)} &= \sum_{j=2^n+1}^{2^{n+1}-2} j(\lambda_j - 2\lambda_{j+1} + \lambda_{j+2}) K_j + 2^{n+1} \\
&\times (\lambda_{2^{n+1}-1} - \lambda_{2^{n+1}}) K_{2^{n+1}} - 2^n (\lambda_{2^n+1} - \lambda_{2^n+2}) K_{2^n} \\
&+ (2\lambda_{2^{n+1}} - \lambda_{2^{n+1}-1}) D_{2^{n+1}} - \lambda_{2^n+1} D_{2^n} \\
&=: \sum_{l=1}^{5} L_{\lambda,n}^{(l)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sup_{N \in \mathbb{N}} |T_{\lambda,2^N}^{(m,k)} a(x)| \leqslant \sum_{l=1}^{5} \sum_{n=0}^{\infty} \bigg| \int_I a(t) \int_0^1 \int_0^1 L_{\lambda,n}^{(l)}(s \dot{+} u) G_n^{(m,k)}(t,s) \\
\times G_n^{(m,-k)}(x,u)\, ds\, du\, dt \bigg|.
\end{aligned}
$$

First let us consider the case $l = 1$ and split the expression into the sums of

$$
A_1(x) := \sum_{n=K}^{\infty} \left| \int_I a(t) \int_0^1 \int_0^1 L_{\lambda,n}^{(1)}(s \dot{+} u) G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)\, ds\, du\, dt \right|
$$

and

$$
A_2(x) := \sum_{n=0}^{K-1} \left| \int_I a(t) \int_0^1 \int_0^1 L_{\lambda,n}^{(1)}(s \dot{+} u) G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)\, ds\, du\, dt \right|.
$$

Using the definition of the atom, (19) and (4) we obtain

$$
\begin{aligned}
A_1(x) &\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \int_0^1 \int_0^1 \bigg| \sum_{l=2^n+1}^{2^{n+1}-2} l(\lambda_l - 2\lambda_{l+1} + \lambda_{l+2}) K_l(s \dot{+} u) \\
&\quad \times G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u) \bigg|\, ds\, du\, dt \\
&\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \int_0^1 \int_0^1 \sum_{j=0}^{n} 2^{j-n} \sum_{i=j}^{n} D_{2^i}(s \dot{+} u \dot{+} 2^{-j-1}) \\
&\quad \times |G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)|\, ds\, du\, dt.
\end{aligned}
$$

By (12) and (1),

$$A_1(x) \leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{2n} \int_I \sum_{\nu=1}^{2^n} \sum_{\mu=1}^{2^n} \int_{(\nu-1)2^{-n}}^{\nu 2^{-n}} \int_{(\mu-1)2^{-n}}^{\mu 2^{-n}} \sum_{j=0}^{n} 2^{j-n} \times \sum_{i=j}^{n} D_{2^i}(s \dot{+} u \dot{+} 2^{-j-1}) q^{2^n|t-\nu 2^{-n}|} q^{2^n|x-\mu 2^{-n}|}\, ds\, du\, dt.$$

Suppose that $\nu < \mu$. It is easy to see that for each ν there exists a set $S_{i,\nu}$ such that

$$D_{2^i}(s \dot{+} u \dot{+} 2^{-j-1}) = \begin{cases} 2^i & \text{if } \mu \in S_{i,\nu}, \\ 0 & \text{if } \mu \notin S_{i,\nu}. \end{cases}$$

Moreover, $|S_{i,\nu}| = 2^{n-i}$ and

$$S_{i,\nu} \subset [\nu + 2^{n-j-1} - 2^{n-i} + 1, \nu + 2^{n-j-1} + 2^{n-i} - 1].$$

This implies

$$\begin{aligned} A_1(x) &\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j \\ &\quad \times \sum_{\nu=1}^{2^n} \sum_{\mu-\nu=2^{n-j-1}-2^{n-i}+1}^{2^{n-j-1}+2^{n-i}-1} q^{2^n|t-\nu 2^{-n}|} q^{2^n|x-\mu 2^{-n}|}\, dt \\ &\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j \\ &\quad \times \sum_{\nu=1}^{2^n} \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|t-\nu 2^{-n}|} q^{2^n|x-2^{-j-1}-\nu 2^{-n}-l2^{-n}|}\, dt \\ &\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|}\, dt, \end{aligned}$$

where we used the inequality

$$\sum_{k=1}^{\infty} q^{|i-k|+|j-k|} \leqslant C(q,r) r^{|i-j|} \qquad (q < r < 1). \tag{21}$$

If

$$A_{1,1}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|}\, dt,$$

$$A_{1,2}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=K}^{n} 2^i \sum_{j=0}^{i} 2^j \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|}\, dt$$

and

$$A_{1,1,1}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \times \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|} 1_{\{2^{-j-1}+8\cdot 2^{K-i} I\}}(x)\, dt,$$

$$A_{1,1,2}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} 2^{-n} \int_I \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \times \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|} 1_{\{2^{-j-1}+8\cdot 2^{K-i} I\}^c}(x)\, dt,$$

then obviously

$$A_1(x) \leqslant A_{1,1}(x) + A_{1,2}(x) \quad \text{and} \quad A_{1,1}(x) \leqslant A_{1,1,1}(x) + A_{1,1,2}(x).$$

It is easy to see that

$$A_{1,1,1}(x) \leqslant C_p 2^{K/p-K} \sum_{n=K}^{\infty} 2^{-n} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j 1_{\{2^{-j-1}+8\cdot 2^{K-i} I\}}(x)$$

and

$$\int_{(16I)^c} |A_{1,1,1}(x)|^p\, dx \leqslant C_p 2^{K(1-p)} \sum_{n=K}^{\infty} 2^{-np} \sum_{i=0}^{K-1} 2^{i(p-1)} \sum_{j=0}^{i} 2^{jp} \leqslant C_p 2^{K(1-2p)} \sum_{i=0}^{K-1} 2^{i(2p-1)} \leqslant C_p,$$

whenever $\frac{1}{2} < p \leqslant 1$.

We conclude that

$$A_{1,1,2}(x) \leqslant C_p 2^{K/p-K} \sum_{n=K}^{\infty} 2^{-n} \sum_{i=0}^{K-1} 2^i \sum_{j=0}^{i} 2^j \times \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{C2^n|x-t_0-2^{-j-1}|} 1_{\{2^{-j-1}+8\cdot 2^{K-i} I\}^c}(x) \leqslant C_p 2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=0}^{n} \sum_{j=0}^{i} 2^j q^{C2^n|x-t_0-2^{-j-1}|}, \tag{22}$$

where t_0 denotes the center of I. Supposing that $x-t_0 \in [2^{-k}, 2^{-k+1})$ for some $1 \leqslant k \leqslant K-1$, we get

$$\begin{aligned}
&2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=0}^{n} \sum_{j=k}^{i} 2^j q^{C2^n|x-t_0-2^{-j-1}|} \\
&\quad \leqslant 2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=0}^{n} \sum_{j=k}^{i} 2^j q^{C2^n|x-t_0|} \\
&\quad \leqslant C2^{K/p-2K} \sum_{n=K}^{\infty} 2^{2n} q^{C2^n|x-t_0|} \\
&\quad \leqslant C2^{K/p-2K}|x-t_0|^{-2},
\end{aligned}$$

because of the inequality

$$\sum_{j=0}^{\infty} 2^{jM} q^{2^j|x-t|} \leqslant C_M|x-t|^{-M} \quad (M>0, x \neq t), \tag{23}$$

which is easy to show, or it can be found in [4,38]. Furthermore,

$$\begin{aligned}
&\int_{(16I)^c} \left| 2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=0}^{n} \sum_{j=k}^{i} 2^j q^{C2^n|x-t_0-2^{-j-1}|} \right|^p dx \\
&\quad \leqslant C_p 2^{K(1-2p)} \int_{(16I)^c} |x-t_0|^{-2p}\, dx \leqslant C_p.
\end{aligned} \tag{24}$$

To investigate the remaining term, observe that

$$\begin{aligned}
&2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=0}^{n\wedge(k-1)} \sum_{j=0}^{(k-1)\wedge i} 2^j q^{C2^n|x-t_0-2^{-j-1}|} \\
&\quad \leqslant 2^{K/p-2K} \sum_{n=K}^{\infty} 2^{n(1+\varepsilon)} \sum_{j=0}^{n\wedge(k-1)} \sum_{i=j}^{n\wedge(k-1)} 2^{(j-n)\varepsilon} 2^{j(1-\varepsilon)} q^{C2^n|x-t_0-2^{-j-1}|} \\
&\quad \leqslant C2^{K/p-2K} \sum_{n=K}^{\infty} 2^{n(1+\varepsilon)} \sum_{j=0}^{k-1} 2^{j(1-\varepsilon)} q^{C2^n|x-t_0-2^{-j-1}|} \\
&\quad \leqslant C2^{K/p-2K} \sum_{j=0}^{k-1} 2^{j(1-\varepsilon)} |x-t_0-2^{-j-1}|^{-(1+\varepsilon)},
\end{aligned}$$

where $0<\varepsilon<1$ is arbitrary and $x-t_0 \in [2^{-k}, 2^{-k+1})$. Moreover, if $k \leqslant n$ then

$$\begin{aligned}
&2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=k}^{n} \sum_{j=0}^{(k-1)\wedge i} 2^j q^{C2^n|x-t_0-2^{-j-1}|} \\
&\quad \leqslant 2^{K/p-2K} \sum_{n=K}^{\infty} 2^{n(1+\varepsilon)} \sum_{j=0}^{k-1} \sum_{i=k}^{n} 2^{(j-n)\varepsilon} 2^{j(1-\varepsilon)} q^{C2^n|x-t_0-2^{-j-1}|}
\end{aligned}$$

$$\leqslant C2^{K/p-2K}\sum_{j=0}^{k-1} 2^{j(1-\varepsilon)}|x-t_0-2^{-j-1}|^{-(1+\varepsilon)}.$$

Hence, if we choose ε such that $(1+\varepsilon)p<1$, then

$$\begin{aligned}
&\int_{(16I)^c}\left|2^{K/p-2K}\sum_{n=K}^{\infty}2^n\sum_{i=0}^{n}\sum_{j=0}^{(k-1)\wedge i}2^j q^{C2^n|x-t_0-2^{-j-1}|}\right|^p dx\\
&\leqslant C_p 2^{K(1-2p)}\sum_{k=1}^{K-1}\sum_{j=0}^{k-1}2^{j(1-\varepsilon)p}\int_{\{x-t_0\in[2^{-k},2^{-k+1})\}}|x-t_0-2^{-j-1}|^{-(1+\varepsilon)p}\,dx\\
&\leqslant C_p 2^{K(1-2p)}\sum_{k=1}^{K-1}\sum_{j=0}^{k-1}2^{j(1-\varepsilon)p}2^{-j(1-(1+\varepsilon)p)}\leqslant C_p. \qquad (25)
\end{aligned}$$

Let us estimate $A_{1,2}(x)$ by the sum of

$$\begin{aligned}
A_{1,2,1}(x) := C_p 2^{K/p}\sum_{n=K}^{\infty}2^{-n}\int_I\sum_{i=K}^{n}2^i\sum_{j=0}^{i}2^j\\
\times\sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|}1_{\{2^{-j-1}+8I\}}(x)\,dt
\end{aligned}$$

and

$$\begin{aligned}
A_{1,2,2}(x) := C_p 2^{K/p}\sum_{n=K}^{\infty}2^{-n}\int_I\sum_{i=K}^{n}2^i\sum_{j=0}^{i}2^j\\
\times\sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|}1_{\{2^{-j-1}+8I\}^c}(x)\,dt.
\end{aligned}$$

Integrating in t we can conclude that

$$\begin{aligned}
A_{1,2,1}(x) &\leqslant C_p 2^{K/p}\sum_{n=K}^{\infty}2^{-n}\sum_{i=K}^{n}2^i\sum_{j=0}^{i}2^j 2^{n-i}2^{-n}1_{\{2^{-j-1}+8I\}}(x)\\
&\leqslant C_p 2^{K/p}\sum_{n=K}^{\infty}2^{-n}\sum_{i=K}^{n}\sum_{j=0}^{i}2^j 1_{\{2^{-j-1}+8I\}}(x).
\end{aligned}$$

It is easy to see that $1_{\{2^{-j-1}+8I\}}(x)=0$ if $x\notin 16I$ and $j\geqslant K$. Henceforth

$$\begin{aligned}
\int_{(16I)^c}|A_{1,2,1}(x)|^p\,dx &\leqslant C_p 2^K\sum_{n=K}^{\infty}2^{-np}\sum_{i=K}^{n}\sum_{j=0}^{K}2^{jp}2^{-K}\\
&\leqslant C_p\sum_{n=K}^{\infty}2^{-(n-K)p}(n-K)\leqslant C_p.
\end{aligned}$$

On the other hand,

$$\begin{aligned}A_{1,2,2}(x) &\leqslant C_p 2^{K/p-K} \sum_{n=K}^{\infty} 2^{-n} \sum_{i=K}^{n} 2^i \sum_{j=0}^{i} 2^j \\ &\quad \times \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{C2^n|x-t_0-2^{-j-1}|} 1_{\{2^{-j-1}+8I\}^c}(x) \\ &\leqslant C_p 2^{K/p-2K} \sum_{n=K}^{\infty} 2^n \sum_{i=0}^{n} \sum_{j=0}^{i} 2^j q^{C2^n|x-t_0-2^{-j-1}|}\end{aligned}$$

and this can be handled in the same way as $A_{1,1,2}(x)$ in (22). This means that we have estimated $A_1(x)$.

Let us consider $A_2(x)$. If $k = m+1$, then for a fixed $s \in [0,1)$, $G_n^{(m,k)}(t,s)$ is constant on I, whenever $n \leqslant K$. Hence $A_2(x) = 0$.

Suppose now that $k \leqslant m$ and set $A(t) := \int_0^t a\, d\lambda$. Integrating by parts we can see that

$$A_2(x) = \sum_{n=0}^{K-1} \left| \int_I A(t) \int_0^1 \int_0^1 L_{\lambda,n}^{(1)}(s \dotplus u) D_t G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)\, ds\, du\, dt \right|.$$

Estimating A_2 in the same way as A_1 we obtain

$$\begin{aligned}A_2(x) &\leqslant C_p 2^{K/p-K} \sum_{n=0}^{K-1} \int_I \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|}\, dt \\ &=: A_{2,1}(x) + A_{2,2},\end{aligned}$$

where

$$\begin{aligned}A_{2,1}(x) &:= C_p 2^{K/p-K} \sum_{n=0}^{K-1} \int_I \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j \\ &\quad \times \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|} 1_{\{2^{-j-1}+8\cdot 2^{K-i}I\}}(x)\, dt, \\ A_{2,2}(x) &:= C_p 2^{K/p-K} \sum_{n=0}^{K-1} \int_I \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j \\ &\quad \times \sum_{l=-2^{n-i}+1}^{2^{n-i}-1} q^{2^n|x-t-2^{-j-1}-l2^{-n}|} 1_{\{2^{-j-1}+8\cdot 2^{K-i}I\}^c}(x)\, dt.\end{aligned}$$

Then

$$A_{2,1}(x) \leqslant C_p 2^{K/p-2K} \sum_{n=0}^{K-1} \sum_{i=0}^{n} 2^i \sum_{j=0}^{i} 2^j 1_{\{2^{-j-1}+8\cdot 2^{K-i}I\}}(x)$$

and

$$\int_{(16I)^c} |A_{2,1}(x)|^p \, dx \leqslant C_p 2^{K(1-2p)} \sum_{n=0}^{K-1} \sum_{i=0}^{n} 2^{i(p-1)} \sum_{j=0}^{i} 2^{jp} \leqslant C_p.$$

For $A_{2,2}$ we have

$$A_{2,2}(x) \leqslant C_p 2^{K/p-2K} \sum_{n=0}^{K-1} 2^n \sum_{i=0}^{n} \sum_{j=0}^{i} 2^j q^{C2^n|x-t_0-2^{-j-1}|}$$

and this was estimated after (22).

Since $|L_{\lambda,n}^{(l)}| \leqslant |L_{\lambda,n}^{(1)}|$ for $l = 2, 3, 4, 5$, the corresponding cases with respect to l can be handled in the same way as above. By interpolation and Theorems 5 and 6 we get the theorem for all $1/2 < p \leqslant 1$. □

If the multiplier λ is piecewise linear then we can prove a stronger result. Let

$$p_{m,k} := \begin{cases} 1/(m-k+2) & \text{if } k \leqslant m, \\ 0 & \text{if } k = m+1. \end{cases}$$

Theorem 8. *Assume that $m \geqslant -1$, $|k| \leqslant m+1$ and $f \in H_p$ with $p_{m,k} < p < \infty$. If (18) is satisfied and*

$$\lambda_j - 2\lambda_{j+1} + \lambda_{j+2} = 0 \quad \textit{for all } j = 2^n + 1, \ldots, 2^{n+1} - 2 \quad (n \in \mathbb{N})$$

then

$$\| \sup_{N \in \mathbb{N}} |T_{\lambda,2^N}^{(m,k)} f| \|_p \leqslant C_p \|f\|_{H_p}.$$

Proof. The proof is similar to that of Theorem 7, so we point out only the main steps. Since $L_{\lambda,n}^{(1)} = 0$ and $D_{2^n} \leqslant K_{2^n}$, it is enough to consider the case according to $l = 3$. We define A_1 and A_2 similarly as in the previous proof. Then

$$\begin{aligned} A_1(x) &\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \int_0^1 \int_0^1 \Big| 2^n (\lambda_{2^n+1} - \lambda_{2^n+2}) K_{2^n}(s \dot{+} u) \\ &\qquad \times G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u) \Big| \, ds \, du \, dt \\ &\leqslant C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \int_0^1 \int_0^1 \sum_{j=0}^{n} 2^{j-n} D_{2^n}(s \dot{+} u \dot{+} 2^{-j-1}) \\ &\qquad \times |G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)| \, ds \, du \, dt. \end{aligned}$$

This means that in the previous proof we should write $i = n$ instead of the sum over i and, moreover, $l = 0$ instead of the sum over l. Since $i = n$, $A_{1,1} = 0$ and

$$A_{1,2}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \sum_{j=0}^{n} 2^j q^{2^n|x-t-2^{-j-1}|} \, dt \leqslant A_{1,2,1}(x) + A_{1,2,2}(x)$$

with

$$A_{1,2,1}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \sum_{j=0}^{n} 2^j q^{2^n|x-t-2^{-j-1}|} 1_{\{2^{-j-1}+8I\}}(x)\, dt,$$

$$A_{1,2,2}(x) := C_p 2^{K/p} \sum_{n=K}^{\infty} \int_I \sum_{j=0}^{n} 2^j q^{2^n|x-t-2^{-j-1}|} 1_{\{2^{-j-1}+8I\}^c}(x)\, dt.$$

Similarly as in the previous proof,

$$\int_{(16I)^c} |A_{1,2,1}(x)|^p \, dx \leqslant C_p 2^K \sum_{n=K}^{\infty} 2^{-np} \sum_{j=0}^{K} 2^{jp} 2^{-K} \leqslant C_p$$

for all $0 < p < 1$. Furthermore, for $r \geqslant 1$,

$$\begin{aligned} A_{1,2,2}(x) &\leqslant C_p 2^{K/p-K} \sum_{n=K}^{\infty} \sum_{j=0}^{n} 2^j q^{C2^n|x-t_0-2^{-j-1}|} 1_{\{2^{-j-1}+8I\}^c}(x) \\ &\leqslant C_p 2^{K/p-(r+1)K} \sum_{n=K}^{\infty} 2^{rn} \sum_{j=0}^{n} 2^j q^{C2^n|x-t_0-2^{-j-1}|}. \end{aligned}$$

Similarly to (24) and (25) we get that

$$\int_{(16I)^c} |A_{1,2,2}(x)|^p \, dx \leqslant C_p$$

for all $1/(r+1) < p < 1/r$. By interpolation we get the inequality for all $1/(r+1) < p \leqslant 1$ and, since $r \geqslant 1$ is arbitrary, for $0 < p \leqslant 1$.

If $k = m+1$, then $A_2(x) = 0$ and the theorem is proved. Suppose that $k \leqslant m$. If

$$A^{(0)} := a, \qquad A^{(j)}(t) := \int_0^t A^{(j-1)} \, d\lambda \quad (j \in \mathbb{N})$$

then

$$\|A^{(j)}\|_\infty \leqslant 2^{K/p-jK} \quad (j \in \mathbb{N}).$$

Integrating by parts $(m-k+1)$-times we obtain

$$\begin{aligned} A_2(x) &= \sum_{n=0}^{K-1} \Bigg| \int_I A^{(m-k+1)}(t) \int_0^1 \int_0^1 2^n (\lambda_{2^n+1} - \lambda_{2^n+2}) K_{2^n}(s \dot{+} u) \\ &\quad \times D_t^{m-k+1} G_n^{(m,k)}(t,s) G_n^{(m,-k)}(x,u)\, ds\, du\, dt \Bigg| \\ &\leqslant 2^{K/p-(m-k+1)K} \sum_{n=0}^{K-1} 2^{(m-k+1)n} \int_I \sum_{j=0}^{n} 2^j q^{2^n|x-t-2^{-j-1}|} \, dt \\ &=: A_{2,1}(x) + A_{2,2}, \end{aligned}$$

where

$$A_{2,1}(x) := 2^{K/p-(m-k+1)K} \sum_{n=0}^{K-1} 2^{(m-k+1)n} \int_I \sum_{j=0}^{n} 2^j q^{2^n|x-t-2^{-j-1}|}$$
$$\times 1_{\{2^{-j-1}+8\cdot 2^{K-n}I\}}(x)\, dt,$$
$$A_{2,2}(x) := 2^{K/p-(m-k+1)K} \sum_{n=0}^{K-1} 2^{(m-k+1)n} \int_I \sum_{j=0}^{n} 2^j q^{2^n|x-t-2^{-j-1}|}$$
$$\times 1_{\{2^{-j-1}+8\cdot 2^{K-n}I\}^c}(x)\, dt.$$

Then the inequality

$$\int_{(16I)^c} |A_2(x)|^p \, dx \leqslant C_p \quad (1/(m-k+2) < p < 1)$$

can be shown by the above methods. This completes the proof of the theorem. □

Note that under the conditions of Theorems 7 or 8 the operator $T_{\lambda,2^N}^{(m,k)}$ is not bounded from L_1 to L_1 in general (see [26]).

Now we are going to extend Theorem 4 to Hardy spaces.

Theorem 9. *If $m \geqslant -1$, $|k| \leqslant m+1$ and λ satisfies the condition in Theorem 7, then*

$$\|Q^{(m,k)}(T_\lambda^{(m,k)} f)\|_p \leqslant C_p \|f\|_{H_p} \quad (f \in H_p)$$

for all $\frac{1}{2} < p < \infty$. If λ fulfills also the condition of Theorem 8, then the inequality holds for all $p_{m,k} < p < \infty$.

Proof. The operators $Q^{(m,k)}$ and $T_\lambda^{(m,k)}$ are bounded on L_p $(1 < p < \infty)$ (see Theorems 4 and 5). Observe that

$$Q^{(m,k)}(T_\lambda^{(m,k)} a)(x) = \left(\sum_{n=0}^{\infty} \left| \int_I a(t) \sum_{j=2^n+1}^{2^{n+1}} \lambda_j c_j^{(m,k)}(t) c_j^{(m,-k)}(x)\, dt \right|^2 \right)^{1/2}$$
$$\leqslant \sum_{n=0}^{\infty} \left| \int_I a(t) \sum_{j=2^n+1}^{2^{n+1}} \lambda_j c_j^{(m,k)}(t) c_j^{(m,-k)}(x)\, dt \right|,$$

where a is a p-atom with support I. The theorem can be shown in the same way as Theorems 7 and 8. □

Since the sequence $(\lambda_j = 1,\ j \in \mathbb{N})$ trivially fulfills the conditions of Theorem 8, we get

Corollary 1. *If $m \geqslant -1$, $|k| \leqslant m+1$ and $p_{m,k} < p < \infty$ then*

$$\|Q^{(m,k)} f\|_p \leqslant C_p \|f\|_{H_p} \quad (f \in H_p).$$

Let us see some other examples for λ, which satisfy the conditions in Theorems 7 and 8. Set

$$\lambda_j^{(1)} := \frac{j-1}{2^n} \quad \text{for} \quad (2^n + 1 \leqslant j \leqslant 2^{n+1}) \quad (n \in \mathbb{N}) \tag{26}$$

and

$$\lambda_j^{(2)} := \frac{2^n}{j-1} \quad \text{for} \quad (2^n + 1 \leqslant j \leqslant 2^{n+1}) \quad (n \in \mathbb{N}). \tag{27}$$

It is easy to see that $\lambda^{(1)}$ satisfies the conditions of Theorems 7 and 8 and, moreover, $\lambda^{(2)}$ fulfills the conditions in Theorem 7. More generally, let $\lambda \in L_\infty([0,\infty))$ be a real function such that for all $n \in \mathbb{N}$

$$\begin{cases} \lambda \text{ is twice continuously differentiable on } (2^n, 2^{n+1}] \text{ except of at most } M \text{ points} \\ \lambda'' \neq 0 \text{ on } (2^n, 2^{n+1}] \text{ except of at most } M \text{ points or intervals,} \\ \text{the function } x \mapsto |x\lambda'(x)| \text{ is bounded where it is defined,} \end{cases}$$

($M \in \mathbb{N}$). Then ($\lambda_n := \lambda(n)$) satisfies the conditions of Theorem 7. Indeed, if $\lambda'' \geqslant 0$ on the interval $(i, j+2) \subset (2^n, 2^{n+1}]$, then λ is convex on this interval and this yields that $\lambda_k - 2\lambda_{k+1} + \lambda_{k+2} \geqslant 0$ for $i \leqslant k \leqslant j$. Hence

$$\sum_{k=i}^{j} k|\lambda_k - 2\lambda_{k+1} + \lambda_{k+2}| = \lambda_i + (i-1)(\lambda_i - \lambda_{i+1}) - j(\lambda_{j+1} - \lambda_{j+2}) - \lambda_{j+1}.$$

By Lagrange's mean value theorem,

$$(i-1)|\lambda_i - \lambda_{i+1}| = (i-1)|\lambda'(\xi(i))| = \frac{i-1}{\xi(i)}|\xi(i)\lambda'(\xi(i))| \leqslant C,$$

where $i < \xi(i) < i+1$.

If $\lambda'' = 0$ at an isolated point u or if λ'' is not twice continuously differentiable at u, $u \in (k, k+1] \subset (2^n, 2^{n+1}]$, then

$$k(\lambda_k - 2\lambda_{k+1} + \lambda_{k+2}) = k(\lambda_k - \lambda_{k+1}) - k(\lambda_{k+1} - \lambda_{k+2}).$$

Applying Lagrange mean value theorem on the intervals (k, u), $(u, k+1)$ and $(k+1, k+2)$, we can see that $k\,|\lambda_k - 2\lambda_{k+1} + \lambda_{k+2}|$ is bounded.

Since on the interval $(2^n, 2^{n+1}]$ there are at most M intervals or isolated points satisfying the above properties, we have shown our assumption.

6. The Sunouchi operator

The following two operators were introduced by Sunouchi [31–33] for Walsh- and trigonometric Fourier series (see also [40, II. p. 224]):

$$U^{(m,k)} f := \left(\sum_{n=0}^{\infty} |s_{2^{n+1}}^{(m,k)} f - \sigma_{2^{n+1}}^{(m,k)} f|^2 \right)^{1/2} \qquad (f \in L_1),$$

$$V^{(m,k)} f := \left(\sum_{n=1}^{\infty} \frac{|s_n^{(m,k)} f - \sigma_n^{(m,k)} f|^2}{n} \right)^{1/2} \quad (f \in L_1).$$

For Walsh–Fourier series Sunouchi [33,32] verified that the operators U and V are bounded on L_p $(1 < p < \infty)$. The analogous statement fails to hold for $p = 1$ (see [34]). However, it was proved by Simon [24] that U is bounded from H_p to L_p for $p = 1$ and by Weisz [37] for all $0 < p \leqslant 1$ (see also [10,25]). In this section these results will be extended to Ciesielski–Fourier series.

Theorem 10. *If $m \geqslant -1$ and $|k| \leqslant m+1$ then*

$$C_p \|V^{(m,k)} f\|_p \leqslant \|U^{(m,k)} f\|_p \leqslant C_p \|V^{(m,k)} f\|_p \tag{28}$$

for $1 < p < \infty$ and

$$\tfrac{1}{3} Q^{(m,k)}(T_{\lambda^{(1)}}^{(m,k)} f) \leqslant U^{(m,k)} f \leqslant Q^{(m,k)}(T_{\lambda^{(1)}}^{(m,k)} f), \tag{29}$$

where the multiplier $\lambda^{(1)}$ was defined in (26).

Proof. With the help of Theorem 3 inequality (28) can be shown in the same way as for Walsh–Fourier series (see [31,33] or [36]).

Observe that

$$s_n^{(m,k)} f(x) - \sigma_n^{(m,k)} f(x) = \sum_{j=2}^{n} \frac{j-1}{n} (f, c_j^{(m,k)}) c_j^{(m,-k)}(x).$$

Let

$$d_{\lambda,n}^{(m,k)} f(x) := \sum_{j=2^n+1}^{2^{n+1}} \lambda_j (f, c_j^{(m,k)}) c_j^{(m,-k)}(x) \quad (n \in \mathbb{N}),$$

$$d_{\lambda}^{(m,k)} f := (d_{\lambda,n}^{(m,k)} f, n \in \mathbb{N}), \quad b_n := 2^{-n-1}, \quad b := (b_n, n \in \mathbb{N}).$$

We will see that the operator $U^{(m,k)}$ can be rewritten as the l_2-norm of the convolution of the two sequences $d_{\lambda^{(1)}}^{(m,k)} f$ and b. Indeed,

$$\begin{aligned}
(d_{\lambda^{(1)}}^{(m,k)} f * b)_n &= \sum_{i=0}^{n} d_{\lambda^{(1)},i}^{(m,k)} f \, b_{n-i} \\
&= \sum_{i=0}^{n} \sum_{j=2^i+1}^{2^{i+1}} \frac{j-1}{2^i} (f, c_j^{(m,k)}) c_j^{(m,-k)} 2^{i-n-1} \\
&= 2^{-n-1} \sum_{j=2}^{2^{n+1}} (j-1)(f, c_j^{(m,k)}) c_j^{(m,-k)} \\
&= s_{2^{n+1}}^{(m,k)} f - \sigma_{2^{n+1}}^{(m,k)} f
\end{aligned}$$

and so

$$U^{(m,k)} f = \|(d^{(m,k)}_{\lambda^{(1)}} f * b)\|_{l_2} \leqslant \|(d^{(m,k)}_{\lambda^{(1)},n} f)\|_{l_2} = Q^{(m,k)}(T^{(m,k)}_{\lambda^{(1)}} f).$$

On the other hand, if $d := (2, -1, 0, 0, \ldots)$, then

$$((s^{(m,k)}_{2^{n+1}} f - \sigma^{(m,k)}_{2^{n+1}} f) * d)_n = 2 \cdot 2^{-n-1} \sum_{j=2}^{2^{n+1}} (j-1)(f, c_j^{(m,k)}) c_j^{(m,-k)} - 2^{-n} \sum_{j=2}^{2^n} (j-1)(f, c_j^{(m,k)}) c_j^{(m,-k)} = d^{(m,k)}_{\lambda^{(1)},n} f$$

and

$$Q^{(m,k)}(T^{(m,k)}_{\lambda^{(1)}} f) \leqslant 3 U^{(m,k)} f$$

which proves the theorem. □

Corollary 2. *If* $m \geqslant -1$, $|k| \leqslant m+1$ *and* $1 < p < \infty$ *then*

$$C_p \|f\|_p \leqslant \|U^{(m,k)} f\|_p \leqslant C_p \|f\|_p \qquad (f \in L_p)$$

and if $p_{m,k} < p \leqslant 1$ *then*

$$\|U^{(m,k)} f\|_p \leqslant C_p \|f\|_{H_p} \quad (f \in H_p). \tag{30}$$

Proof. The right-hand side of the inequalities follow from Theorem 4, 5 and 9. For the left hand side observe that $\lambda^{(2)} = (\lambda^{(1)})^{-1}$ and hence

$$\begin{aligned}\|f\|_p &= \|T^{(m,k)}_{\lambda^{(2)}}(T^{(m,k)}_{\lambda^{(1)}} f)\|_p \leqslant C_p \|T^{(m,k)}_{\lambda^{(1)}} f\|_p \\ &\leqslant C_p \|Q^{(m,k)}(T^{(m,k)}_{\lambda^{(1)}} f)\|_p \leqslant C_p \|U^{(m,k)} f\|_p.\end{aligned}$$

The proof of the corollary is complete. □

Note that the converse inequality to (30) for Walsh- and Walsh–Kaczmarz series was verified by Daly and Phillips [10] and Simon [25,27,28].

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